Periodic asymptotic dynamics of the measure solutions to an equal mitosis equation

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Abstract

We are interested in a non-local partial differential equation modeling equal mitosis and we prove that the solutions, in suitable spaces of weighted signed measures, present persistent asymptotic oscillations. To do so we adopt a duality approach, which is also well suited for proving the well-posedness when the division rate is unbounded. The main difficulty for characterizing the asymptotic behavior is to define the projection onto the subspace of periodic (rescaled) solutions. We achieve this by using the generalized relative entropy structure of the dual problem. The second main difficulty is to estimate the speed of convergence to the oscillating behavior. We use Harris's ergodic theorem on sub-problems to get an explicit exponential rate of convergence, which is a major novelty.

Keywords: growth-fragmentation equation, self-similar fragmentation, measure solutions, long-time behavior, general relative entropy, Harris's theorem, periodic semigroups

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1 Introdution

We are interested in the following nonlocal transport equation

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}\big(x\,u(t,x)\big) + B(x)u(t,x) = 4B(2x)u(t,2x), \qquad x > 0. \tag{1}$$

It appears as an idealized size-structured model for the bacterial cell division cycle [7, 56], and it is an interesting and challenging critical case of the general

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linear growth-fragmentation equation, as we will explain below. The unknown $u(t,x)$ represents the population density of cells of size x at time t, which evolves according to two phenomena: the individual exponential growth which results in the transport term $\partial_x(xu(t,x))$, and the equal mitosis corresponding to the nonlocal infinitesimal term $2B(2x)u(t,2x)2\mathrm{d}x - B(x)u(t,x)\mathrm{d}x$.

Equation (1) is also the Kolmogorov (forward) equation of the underlying piecewise deterministic branching process [19, 23, 25, 29, 47]. Let us explain this briefly and informally. Consider the measure-valued branching process $(Z_t)_{t\geqslant 0}$ defined as the empirical measure

$$Z_t = \sum_{i\in V_t} \delta_{X_t^i}$$

where V_t is the set of individuals alive at time t and $\{X_t^i : i \in V_t\}$ the set of their sizes. For each individual $i \in V_t$ the size X_t^i grows exponentially fast following the deterministic flow $\frac{\mathrm{d}}{\mathrm{d}t}X_t^i = X_t^i$ until a division time T_i which occurs stochastically in a Poisson-like fashion with rate $B(X_t^i)$. Then the individual i dies and gives birth to two daughter cells i_1 and i_2 with size $X_{T_i}^{i_1} = X_{T_i}^{i_2} = \frac{1}{2}X_{T_i}^i$. Taking the expectancy of the random measures Z_t, we get a family of measures $u(t,\cdot)$ defined for any Borel set $A \subset (0,\infty)$ by

$$u(t,A) = \mathbb{E}[Z_t(A)] = \mathbb{E}\big[\#\{i \in V_t : X_t^i \in A\}\big],$$

which is a weak solution to the Kolmogorov Equation (1).

Another Kolmogorov equation is classically associated to $(Z_t)_{t\geqslant 0}$, which is the dual equation of (1)

$$\frac{\partial}{\partial t}\varphi(t,x) = x\frac{\partial}{\partial x}\varphi(t,x) + B(x)\big[2\varphi(t,x/2) - \varphi(t,x)\big], \qquad x > 0. \tag{2}$$

This second equation is sometimes written in its backward version where $\frac{\partial}{\partial t}\varphi(t,x)$ is replaced by $-\frac{\partial}{\partial t}\varphi(t,x)$, and is then usually called the Kolmogorov backward equation. Nevertheless since the division rate $B(x)$ does not depend on time, we prefer here writing this backward equation in a forward form. Indeed in this case we have that for an observation function f,

$$\varphi(t,x) := \mathbb{E}[Z_t(f)\,|\,Z_0 = \delta_x] := \mathbb{E}\bigg[\sum_{i\in V_t} f(X_t^i)\,\Big|\,Z_0 = \delta_x\bigg]$$

is the solution to (2) with initial data $\varphi(0,x) = f(x)$.

Equation (1) is then naturally defined on a space of measure, while Equation (2) is defined on a space of functions.

As we already mentioned, Equation (1) is a particular case of the growth-fragmentation equation which reads in its general form

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}\big(g(x)u(t,x)\big) + B(x)u(t,x) = \int_x^\infty k(y,x)B(y)u(t,y)\mathrm{d}y.$$

In this model the deterministic growth flow is given by $\frac{\mathrm{d}x}{\mathrm{d}t} = g(x)$ and the mitosis $x \to \frac{x}{2}$ is replaced by the more general division $x \to y < x$ with a kernel $k(x, \mathrm{d}y)$. Equation (1) then corresponds to the case $g(x) = x$ and $k(x, \mathrm{d}y) = 2\delta_{y=\frac{x}{2}}$. The long time behavior of the growth-fragmentation equation is strongly related to the existence of steady size distributions, namely solutions of the form $\mathcal{U}(x)\mathrm{e}^{\lambda t}$ with $\mathcal{U}$ nonnegative and integrable. It is actually equivalent to say that $\mathcal{U}$ is a Perron eigenfunction associated to the eigenvalue λ. Such an eigenpair $(\lambda, \mathcal{U})$ typically exists when, roughly speaking, the fragmentation rate B dominates the growth speed g at infinity and on the contrary g dominates B around the origin (see [27, 28, 30] for more details). In most cases where this existence holds, the solutions behave asymptotically like the steady size distribution $\mathcal{U}(x)\mathrm{e}^{\lambda t}$. This property, known as asynchronous exponential growth [58], has been proved by many authors using various methods since the pioneering work of Diekmann, Heijmans, and Thieme [26]. Most of these results focus on one of the two special cases $g(x) = 1$ (linear individual growth) or $g(x) = x$ (exponential individual growth). When $g(x) = 1$ it has been proved for the equal mitosis or more general kernels $k(x, y)$ by means of spectral analysis of semigroups [9, 26, 38, 42, 50], general relative entropy method [24, 49] and/or functional inequalities [2, 18, 44, 51, 52, 53], Doeblin's type condition [3, 14, 16, 55] combined with the use of Lyapunov functions [5, 13], coupling arguments [6, 22, 45], many-to-one formula [23], or explicit expression of the solutions [59]. For the case $g(x) = x$ asynchronous exponential growth is proved under the assumption that $k(x, \mathrm{d}y)$ has an absolutely continuous part with respect to the Lebesgue measure: by means of spectral analysis of semigroups [9, 42, 50], general relative entropy method [32, 49] and/or functional inequalities [2, 17, 18, 37], Foster-Lyapunov criteria [13], Feynman-Kac [10, 12, 11, 21] or many-to-one formulas [46].

The assumption that the fragmentation kernel has a density part when $g(x) = x$ is a crucial point, not only a technical restriction. In the equal mitosis case of Equation (1) for instance, asynchronous exponential growth does not hold. It can be easily understood through the branching process $(Z_t)_{t\geqslant 0}$. If at time $t = 0$ the population is composed of only one individual with deterministic size $x > 0$, then for any positive time t and any $i \in V_t$ we have that $X_t^i \in \{x\mathrm{e}^t 2^{-k} : k \in \mathbb{N}\}$. This observation was made already by Bell and Anderson in [7] and it has two important consequences.

First the solution $u(t, \cdot) = \mathbb{E}[Z_t]$ cannot relax to a steady size distribution and it prevents Equation (1) from having the asynchronous exponential growth property. The dynamics does not mix enough the trajectories to generate ergodicity, and the asymptotic behavior keeps a strong memory of the initial data. This situation has been much less studied than the classical ergodic case. In [26, 43] Diekmann, Heijmans, and Thieme made the link with the existence of a nontrivial boundary spectrum: all the complex numbers $1 + \frac{2ik\pi}{\log 2}$, with k lying in $\mathbb{Z}$, are eigenvalues. As a consequence the Perron eigenvalue $\lambda = 1$ is not strictly dominant and it results in persistent oscillations, generated by the boundary eigenfunctions. The convergence to this striking behavior was first proved in [38] in the space $L^1([\alpha, \beta])$ with $[\alpha, \beta] \subset (0, \infty)$. More recently it has been obtained in $L^1(0,\infty)$ for monomial division rates and smooth initial data [57], and in $L^2((0,\infty), x/\mathcal{U}(x)\,\mathrm{d}x)$ [8].

Second, it highlights the lack of regularizing effect of the equation. If the initial distribution is a Dirac mass, then the solution is a Dirac comb for any time. It contrasts with the cases of density fragmentation kernels for which the singular part of the measure solutions vanishes asymptotically when times goes to infinity [24], and gives an additional motivation for studying Equation (1) in a space of measures.

The aim of the present paper is to prove the convergence to asymptotic oscillations for the measure solutions of Equation (1).

Measure solutions to structured populations dynamics PDEs have attracted increasing attention in the last few years, and there exist several general well-posedness results [15, 20, 33, 34, 39]. However they do not apply here due to the unboundedness of the function B, which is required for the Perron eigenfunction $\mathcal{U}$ to exist, see Section 2.2. We overcome this difficulty by adopting a duality approach in the spirit of [4, 5, 31, 35], which is also convenient for investigating the long time behavior.

In [38] Greiner and Nagel deduce the convergence from a general result of spectral theory of positive semigroups, valid in L^p spaces with $1 \leqslant p < \infty$ [1, C-IV, Th. 2.14]. To be able to apply this abstract result they need to consider on a compact size interval $[\alpha, \beta]$. In [57] van Brunt *et al.* take advantage of the Mellin transform to solve Equation (1) explicitly and deduce the convergence in $L^1(0,\infty)$. But this method requires the division rate to be monomial, namely $B(x) = x^r$ with $r > 0$, and $u(0,\cdot)$ to be a $\mathcal{C}^2$ function with polynomial decay at 0 and ∞. In [8] the authors combine General Relative Entropy inequalities and the Hilbert structure of the space $L^2((0,\infty), x/\mathcal{U}(x)\,\mathrm{d}x)$ to prove that the solutions converge to their orthogonal projection onto the closure of the subspace spanned by the boundary eigenfunctions. The general relative entropy method has been recently extended to the measure solutions of the growth-fragmentation equation with smooth fragmentation kernel [24], but this cannot be applied to the singular case of the mitosis kernel. Our approach rather relies on the general relative entropy of the dual equation (2). It allows us to both define a projector on the boundary eigenspace despite the absence of Hilbert structure and prove the weak-* convergence to this projection. We then use Harris's ergodic theorem to strengthen it into a convergence in weighted total variation norm with exponential speed. Besides, the exponential rate of convergence can be estimated explicitly in terms of the division rate B.

The paper is organized as follows. In the next section, we state our main result. In Section 3, we prove the well-posedness of Equation (1) in the framework of measure solutions. Section 4 is devoted to the analysis of the long time asymptotic behavior. Finally, in a last section, we draw some future directions that can extend the present work.

2 Preliminaries and the main result

Before stating our main result, we introduce the space of weighted signed measures in which we will work and we recall existing spectral results about Equation (1).

2.1 Weighted signed measures and measure solutions

A particular feature of Equation (1) is the exponential growth of the total mass. Indeed, a formal integration against the measure $x\,\mathrm{d}x$ over $(0,\infty)$ leads to the balance law

$$\int_0^\infty xu(t,x)\mathrm{d}x = \mathrm{e}^t \int_0^\infty xu(0,x)\mathrm{d}x.$$

Due to this property, the weighted Lebesgue space $L^1((0,\infty), x\,\mathrm{d}x)$ provides a natural framework for studying Equation (1). In the measure solutions framework, the space $\mathcal{M}$ of finite signed Borel measures on $(0,\infty)$ extends the space $L^1((0,\infty),\mathrm{d}x)$. Thus a possible choice for the setting of our work could be the subspace

$$\Big\{\mu \in \mathcal{M},\ \int_0^\infty x\,|\mu|(\mathrm{d}x) < \infty\Big\}$$

where the positive measure $|\mu|$ is the total variation of μ, see [54] for instance. However this subspace of $\mathcal{M}$ does not contain $L^1((0,\infty), x\,\mathrm{d}x)$, so we prefer to define a more relevant *ad hoc* space.

For a weight function $w : (0,\infty) \to (0,\infty)$, we denote by $\mathcal{M}_+(w)$ the cone of positive measures μ on $(0,\infty)$ such that

$$\int_{(0,\infty)} w\,\mathrm{d}\mu < \infty.$$

We define the space of *weighted signed measures* $\mathcal{M}(w)$ as the quotient space

$$\mathcal{M}(w) := {}^{\textstyle \mathcal{M}_+(w) \times \mathcal{M}_+(w)}\big/_{\textstyle\sim}$$

where $(\mu_1,\mu_2) \sim (\tilde\mu_1,\tilde\mu_2)$ if $\mu_1 + \tilde\mu_2 = \tilde\mu_1 + \mu_2$. Clearly $\mathcal{M}(w)$ is isomorphic to $\mathcal{M}$ through the canonical mapping

$$\left\{\begin{array}{ccc} \mathcal{M}(w) & \to & \mathcal{M} \\ \mu & \mapsto & \Big\{A \mapsto \displaystyle\int_A w\,\mathrm{d}\mu_1 - \int_A w\,\mathrm{d}\mu_2\Big\} \end{array}\right. \tag{3}$$

where (μ_1,μ_2) is any representative of the equivalence class μ, and this motivates the notation $\mu = \mu_1 - \mu_2$. Through this isomorphism the Hahn-Jordan decomposition of signed measures ensures that for any $\mu \in \mathcal{M}(w)$ there exists a unique couple $(\mu_+,\mu_-) \in \mathcal{M}_+(w) \times \mathcal{M}_+(w)$ of mutually singular measures such that $\mu = \mu_+ - \mu_-$, and we can define its total variation $|\mu| := \mu_+ + \mu_-$. We endow $\mathcal{M}(w)$ with the weighted total variation norm

$$\|\mu\|_{\mathcal{M}(w)} := \int_{(0,\infty)} w\,\mathrm{d}|\mu| = \int_{(0,\infty)} w\,\mathrm{d}\mu_+ + \int_{(0,\infty)} w\,\mathrm{d}\mu_-$$

which makes it a Banach space, the isomorphism (3) being actually an isometry if $\mathcal{M}$ is endowed with the standard total variation norm.

Notice that if w is not bounded from below by a positive constant, there are elements of $\mathcal{M}(w)$ that are not strictly speaking measures. There exist some $\mu = \mu_1 - \mu_2$ and Borel sets $A \subset (0,\infty)$ such that $\mu_1(A) = \mu_2(A) = +\infty$, so that $\mu(A)$ does not make sense. Nevertheless, the isomorphism (3) ensures that it becomes a measure once multiplied by the weight function w, and this motivates calling it a weighted signed measure. Another motivation is the analogy with weighted L^1 spaces: we can naturally associate to a function $f \in L^1((0,\infty), w(x)\,\mathrm{d}x)$ the weighted measure $\mu(\mathrm{d}x) = f_+(x)\mathrm{d}x - f_-(x)\mathrm{d}x$, thus defining a canonical injection of $L^1((0,\infty), w(x)\,\mathrm{d}x)$ into $\mathcal{M}(w)$.

Now that the convenient space $\mathcal{M}(w)$ is defined, we give some useful properties of its natural action on measurable functions. Denote by $\mathcal{B}(w)$ the space of Borel functions $f : (0,\infty) \to \mathbb{R}$ such that the quantity

$$\|f\|_{\mathcal{B}(w)} := \sup_{x>0} \frac{|f(x)|}{w(x)} \tag{4}$$

is finite. An element μ of $\mathcal{M}(w)$ defines a linear form on $\mathcal{B}(w)$ through

$$\mu(f) := \int_{(0,\infty)} f\,\mathrm{d}\mu_+ - \int_{(0,\infty)} f\,\mathrm{d}\mu_-.$$

We also define the subset $\mathcal{C}(w) \subset \mathcal{B}(w)$ of continuous functions, and the subset $\mathcal{C}_0(w) \subset \mathcal{C}(w)$ of the functions such that the ratio $f(x)/w(x)$ vanishes at zero and infinity. The isomorphism (3) combined with the Riesz representation theorem, which states that $\mathcal{M} \simeq \mathcal{C}_0(0,\infty)'$, ensures that $\mathcal{M}(w) \simeq \mathcal{C}_0(w)'$ with the identity

$$\|\mu\|_{\mathcal{M}(w)} = \sup_{\|f\|_{\mathcal{B}(w)} \leqslant 1} \mu(f)$$

where the supremum is taken over $\mathcal{C}_0(w)$. Actually the supremum can also be taken over $\mathcal{B}(w)$ and in this case it is even a maximum (take $f(x) = w(x)$ on the support of μ_+ and $f(x) = -w(x)$ on the support of μ_-).

The case $w(x) = x$ plays a special role in our study and it will be convenient to denote by $\dot{\mathcal{M}}, \dot{\mathcal{B}}, \dot{\mathcal{C}}, \dot{\mathcal{C}}_0$ the spaces $\mathcal{M}(w), \mathcal{B}(w), \mathcal{C}(w), \mathcal{C}_0(w)$ corresponding to this choice of weight function. Since the boundary eigenvalues are complex, it is also useful to consider the space $\dot{\mathcal{C}}^{\mathbb{C}}$ of continuous functions $f : (0,\infty) \to \mathbb{C}$ such that $\|f\|_{\dot{\mathcal{C}}} < \infty$, where the norm $\|\cdot\|_{\dot{\mathcal{C}}}$ is still defined by (4) but with $|\cdot|$ denoting the modulus instead of the absolute value, as well as the space of weighted complex measures $\dot{\mathcal{M}}^{\mathbb{C}} := \dot{\mathcal{M}} + i\dot{\mathcal{M}}$. The action of $\dot{\mathcal{M}}^{\mathbb{C}}$ of $\dot{\mathcal{C}}^{\mathbb{C}}$ is naturally defined by

$$\mu(f) := (\mathrm{Re}\mu)(\mathrm{Re}f) - (\mathrm{Im}\mu)(\mathrm{Im}f) + i\big[(\mathrm{Re}\mu)(\mathrm{Im}f) + (\mathrm{Im}\mu)(\mathrm{Re}f)\big].$$

It remains to define a notion of solutions in the space $\dot{\mathcal{M}}$ for Equation (1). Let us first define the operator $\mathcal{A}$ acting on the space $\mathcal{C}^1(0,\infty)$ of continuously differentiable functions via

$$\mathcal{A}f(x) := xf'(x) + B(x)\big(2f(x/2) - f(x)\big)$$

and its domain

$$D(\mathcal{A}) = \big\{f \in \dot{\mathcal{B}} \cap \mathcal{C}^1(0,\infty) : \mathcal{A}f \in \dot{\mathcal{B}}\big\}.$$

With this notation the dual equation (2) simply reads

$$\partial_t \varphi = \mathcal{A}\varphi.$$

The definition we choose for the measure solutions to Equation (1) is of the "mild" type in the sense that it relies on an integration in time, and of the "weak" type in the sense that it involves test functions in space.

Definition 1. *A family $(\mu_t)_{t\geqslant 0} \subset \dot{\mathcal{M}}$ is called a measure solution to Equation (1) if for all $f \in \dot{\mathcal{C}}$ the mapping $t \mapsto \mu_t f$ is continuous, and for all $t \geqslant 0$ and all $f \in D(\mathcal{A})$*

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(\mathcal{A}f)\,\mathrm{d}s. \tag{5}$$

2.2 Dominant eigenvalues and periodic solutions

As we already mentioned in the introduction, the long term behavior of Equation (1) – as well as that of Equation (2) – is strongly related to the associated Perron eigenvalue problem, which consists in finding a constant λ together with nonnegative and nonzero $\mathcal{U}$ and ϕ such that

$$\big(x\mathcal{U}(x)\big)' + (B(x) + \lambda)\,\mathcal{U}(x) = 4B(2x)\mathcal{U}(2x), \tag{6}$$

$$-\,x\phi'(x) + (B(x) + \lambda)\,\phi(x) = 2B(x)\phi\Big(\frac{x}{2}\Big). \tag{7}$$

This problem has been solved under various assumptions on the division rate B in [28, 32, 41, 48]. The most general result is the one obtained as a particular case of [28, Theorem 1], which supposes that B satisfies the following conditions

$$\begin{cases} B : (0,\infty) \to [0,\infty) \text{ is locally integrable,} \\ \operatorname{supp} B = [b, +\infty) \text{ for some } b \geqslant 0, \\ \exists b_0, \gamma_0, K_0 > 0,\ \forall x < b_0, \quad B(x) \leqslant K_0 x^{\gamma_0} \\ \exists b_1, \gamma_1, \gamma_2, K_1, K_2 > 0\ \forall x > b_1 \quad K_1 x^{\gamma_1} \leqslant B(x) \leqslant K_2 x^{\gamma_2}. \end{cases} \tag{8}$$

Theorem ([28]). *Under assumption* (8)*, there exists a unique nonnegative eigenfunction* $\mathcal{U} \in L^1(0,\infty)$ *solution to* (6) *and normalized by* $\int_0^\infty x\,\mathcal{U}(x)\mathrm{d}x = 1$*. It is associated to the eigenvalue* $\lambda = 1$ *and to the adjoint eigenfunction* $\phi(x) = x$ *solution to* (7)*. Moreover,*

$$\forall r \in \mathbb{R}, \qquad x^r\,\mathcal{U} \in L^1(0,\infty) \cap L^\infty(0,\infty).$$

As already noticed in [26] (see also example 2.15, p.354 in [1]), the Perron eigenvalue $\lambda = 1$ is not strictly dominant in the present case. There is an infinite number of (complex) eigenvalues with real part equal to 1. More precisely for all $k \in \mathbb{Z}$ the triplet $(\lambda_k, \mathcal{U}_k, \phi_k)$ defined from $(\lambda, \mathcal{U}, \phi)$ by

$$\lambda_k = 1 + \frac{2ik\pi}{\log 2}, \qquad \mathcal{U}_k(x) = x^{-\frac{2ik\pi}{\log 2}}\mathcal{U}(x), \qquad \phi_k(x) = x^{1+\frac{2ik\pi}{\log 2}},$$

verifies (6)-(7). In such a situation the asynchronous exponential growth property cannot hold, since for any $k \in \mathbb{Z} \setminus \{0\}$ the functions

$$\mathrm{Re}\big(\mathcal{U}_k(x)\mathrm{e}^{\lambda_k t}\big) = \\ \Big[\cos\Big(\frac{2k\pi}{\log 2}\log x\Big)\cos\Big(\frac{2k\pi}{\log 2}t\Big) - \sin\Big(\frac{2k\pi}{\log 2}\log x\Big)\sin\Big(\frac{2k\pi}{\log 2}t\Big)\Big]\mathcal{U}(x)\mathrm{e}^t$$

and

$$\mathrm{Im}\big(\mathcal{U}_k(x)\mathrm{e}^{\lambda_k t}\big) = \\ \Big[\cos\Big(\frac{2k\pi}{\log 2}\log x\Big)\sin\Big(\frac{2k\pi}{\log 2}t\Big) - \sin\Big(\frac{2k\pi}{\log 2}\log x\Big)\cos\Big(\frac{2k\pi}{\log 2}t\Big)\Big]\mathcal{U}(x)\mathrm{e}^t$$

are solutions to Equation (1) that oscillate around $\mathcal{U}(x)\mathrm{e}^t$.

2.3 Statement of the main result

In [8] it is proved that the family $(\mathcal{U}_k(x)\mathrm{e}^{\lambda_k t})_{k\in\mathbb{Z}}$ of solutions is enough to get the long time behavior of all the others in the space $L^2((0,\infty), x/\mathcal{U}(x)\,\mathrm{d}x)$. More precisely it is proved that

$$\Big\| u(t,\cdot)\mathrm{e}^{-t} - \sum_{k\in\mathbb{Z}} (u(0,\cdot), \mathcal{U}_k)\,\mathcal{U}_k\,\mathrm{e}^{(\lambda_k - 1)t} \Big\|_{L^2((0,\infty),\, x/\mathcal{U}(x)\,\mathrm{d}x)} \xrightarrow[t\to+\infty]{} 0 \qquad (9)$$

where $(\cdot\,,\cdot)$ stands for the canonical inner product of the complex Hilbert space $L^2((0,\infty), x/\mathcal{U}(x)\,\mathrm{d}x)$. Such an oscillating behavior also occurs in a L^1 setting, as shown by [26, 38, 57]. But the techniques used in these papers (abstract theory of semigroups or Mellin transform) do not allow to make appear explicitly the eigenelements $(\lambda_k, \mathcal{U}_k, \phi_k)$ in the limiting dynamic equilibrium. The objective of the present paper is threefold: prove the well-posedness of Equation (1) in $\dot{\mathcal{M}}$, extend the previous results about the long time behavior to this large space, and characterize the oscillating limit in terms of $(\lambda_k, \mathcal{U}_k, \phi_k)$.

To prove the existence and uniqueness of solutions to Equation (1) in the sense of Definition 1, we make the following assumption on the division rate:

$$B : (0,\infty) \to [0,\infty) \text{ is continuous and bounded around } 0. \tag{10}$$

Since we work in $\dot{\mathcal{M}}$, it is convenient to define the family of complex measures $\nu_k \in \dot{\mathcal{M}}^{\mathbb{C}}$ with Lebesgue density $\mathcal{U}_k$, *i.e.*

$$\nu_k(\mathrm{d}x) = \mathcal{U}_k(x)\,\mathrm{d}x.$$

The following theorem summarizes the main results of the paper.

Theorem 1. *Let $\mu_0 \in \dot{\mathcal{M}}$. If Assumption* (10) *is verified, then there exists a unique measure solution $(\mu_t)_{t\geqslant 0}$ to Equation* (1) *in the sense of Definition 1. If B satisfies additionally* (8)*, then there exists a unique* $\log 2$*-periodic family $(\rho_t)_{t\geqslant 0} \subset \dot{\mathcal{M}}$ such that for all $f \in \dot{\mathcal{C}}_0$*

$$\mathrm{e}^{-t}\mu_t(f) - \rho_t(f) \xrightarrow[t\to+\infty]{} 0.$$

Moreover, for any $t \geqslant 0$, the weighted measure ρ_t is characterized through a Fejér type sum: for all $f \in \mathcal{C}_c^1(0,\infty)$

$$\rho_t(f) = \lim_{N\to\infty} \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N}\Big)\mu_0(\phi_k)\nu_k(f)\mathrm{e}^{\frac{2ik\pi}{\log 2}t}.$$

Finally, consider two real numbers r_1 and r_2 with $r_1 < 1 < r_2$ and define the weight $w(x) = x^{r_1} + x^{r_2}$. If μ_0 belongs to $\mathcal{M}(w)$ then so does ρ_0 and there exist computable constants $C \geqslant 1$ and $a > 0$, that depend only on r_1, r_2 and B, such that for all $t \geqslant 0$

$$\left\|\mathrm{e}^{-t}\mu_t - \rho_t\right\|_{\mathcal{M}(w)} \leqslant C\mathrm{e}^{-at}\left\|\mu_0 - \rho_0\right\|_{\mathcal{M}(w)}.$$

Let us make some comments about these results:

(i) It is worth noticing that the well-posedness of Equation (1) do not require any upper bound for the division rate. It contrasts with existing results in Lebesgue spaces where at most polynomial growth is usually assumed.

(ii) In [8, 38, 57] the convergence to the oscillating behavior is proved to occur in norm but without any estimate on the speed. Here we extend the convergence to measure solutions and provide for the first time an explicit rate of decay in suitable weighted total variation norm.

(iii) In (9) the dynamic equilibrium is characterized as a Fourier type series. In our result it is replaced by a Fejér sum, namely the Cesàro means of the Fourier series.

(iv) Even though all the ν_k have a density with respect to the Lebesgue measure, the limit ρ_t does not in general. Indeed, as noticed in the introduction, if for instance $\mu_0 = \delta_x$ then $\operatorname{supp}\mu_t \subset \{x\mathrm{e}^t 2^{-k} : k \in \mathbb{N}\}$, and consequently $\operatorname{supp}\rho_t \subset \{x\mathrm{e}^t 2^{-k} : k \in \mathbb{Z}\}$ and it is thus a Dirac comb.

(v) We easily notice in the explicit formula of ρ_t that if μ_0 is such that $\mu_0(\phi_k) = 0$ for all $k \neq 0$, then there is no oscillations and the solution behaves asymptotically like $\mathcal{U}(x)\mathrm{e}^t$, similarly to the asynchronous exponential growth case. Such initial distributions actually do exist, as for instance the one proposed in [57] which reads in our setting

$$\mu_0(\mathrm{d}x) = \frac{1}{x^2}\mathbb{1}_{[1,2]}(x)\,\mathrm{d}x$$

where $\mathbb{1}_{[1,2]}$ denotes the indicator function of the interval $[1,2]$.

3 Well-posedness in the measure setting

Our method consists in two main steps. In the first place, we prove the well-posedness of Equation (2) by means of fixed point techniques. Afterwards, we combine this result and a duality property to define the unique solution to Equation (1).

3.1 The dual equation

We actually prove slightly more than the well-posedness of Equation (2) in $\dot{\mathcal{B}}$. Let us first introduce some useful functional spaces. For a subset $\Omega \subset \mathbb{R}^d$, we denote by $\mathcal{B}_{loc}(\Omega)$ the space of functions $f : \Omega \to \mathbb{R}$ that are bounded on $\Omega \cap B(0,r)$ for any $r > 0$, and by $\mathcal{B}(\Omega)$ the (Banach) subspace of bounded functions endowed with the supremum norm $\|f\|_\infty = \sup_{x\in\Omega}|f(x)|$. Using these spaces allows us to prove the well-posedness without needing any upper bound at infinity on the division rate B.

In the following proposition, we prove that for any $f \in \mathcal{B}_{loc}(0,\infty)$ there exists a unique solution $\varphi \in \mathcal{B}_{loc}([0,\infty)\times(0,\infty))$ to Equation (2) in a mild sense (Duhamel formula) with initial condition $\varphi(0,\cdot) = f$. Moreover, we show that if $f \in \mathcal{C}^1(0,\infty)$ then φ is also continuously differentiable and verifies Equation (2) in the classical sense.

Proposition 2. *Assume that B satisfies* (10). *Then for any $f \in \mathcal{B}_{loc}(0,\infty)$ there exists a unique $\varphi \in \mathcal{B}_{loc}([0,\infty)\times(0,\infty))$ such that for all $t \geqslant 0$ and $x > 0$*

$$\varphi(t,x) = f(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s} + 2\int_0^t B(x\mathrm{e}^\tau)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s}\varphi\Big(t-\tau, \frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau.$$

Moreover if f is nonnegative/continuous/continuously differentiable, then so is φ. In the latter case φ verifies for all $t, x > 0$

$$\frac{\partial}{\partial t}\varphi(t,x) = \mathcal{A}\varphi(t,\cdot)(x) = x\frac{\partial}{\partial x}\varphi(t,x) + B(x)\big(2\varphi(t,x/2) - \varphi(t,x)\big).$$

Proof. Let $f \in \mathcal{B}_{loc}(0,\infty)$ and define on $\mathcal{B}_{loc}([0,\infty)\times(0,\infty))$ the mapping Γ by

$$\Gamma g(t,x) = f(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\,\mathrm{d}s} + 2\int_0^t B(x\mathrm{e}^\tau)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\,\mathrm{d}s} g\Big(t-\tau, \frac{x\mathrm{e}^\tau}{2}\Big)\,\mathrm{d}\tau.$$

For $T,K>0$ define the set $\Omega_{T,K} = \{(t,x)\in[0,T]\times(0,\infty),\ x\mathrm{e}^t < K\}$. Clearly Γ induces a mapping $\mathcal{B}(\Omega_{T,K}) \to \mathcal{B}(\Omega_{T,K})$, still denoted by Γ. To build a fixed point of Γ in $\mathcal{B}_{loc}([0,\infty)\times(0,\infty))$ we prove that it admits a unique fixed point in any $\mathcal{B}(\Omega_{T,K})$, denoted $\varphi_{T,K}$, that we will build piecewisely on subsets of $\Omega_{T,K}$.

Let $K>0$ and $t_0 < 1/(2\sup_{(0,K)} B)$. For any $g_1, g_2 \in \mathcal{B}(\Omega_{t_0,K})$ we have

$$\|\Gamma g_1 - \Gamma g_2\|_\infty \leqslant 2t_0 \sup_{(0,K)} B\ \|g_1 - g_2\|_\infty$$

and Γ is a contraction. The Banach fixed point theorem then guarantees the existence of a unique fixed point $\varphi_{t_0,K}$ of Γ in $\mathcal{B}(\Omega_{t_0,K})$. The first step to construct the unique fixed point of Γ on $\mathcal{B}(\Omega_{T,K})$ is to set $\varphi_{T,K|\Omega_{t_0,K}} := \varphi_{t_0,K}$. We can repeat this argument on $\mathcal{B}(\Omega_{t_0,K\mathrm{e}^{-t}})$ with f being replaced by $\varphi_{t_0,K}(t_0,\cdot)$ to obtain a fixed point $\varphi_{t_0,K\mathrm{e}^{-t}}$. Then we set $\varphi_{T,K|\Omega_{t_0,K\mathrm{e}^{-t}}}(\cdot+t_0,\cdot) := \varphi_{t_0,K\mathrm{e}^{-t}}$, thus defining $\varphi_{T,K}$ on $\Omega_{2t_0,K}$. Iterating the procedure we finally get a unique fixed point $\varphi_{T,K}$ of Γ in $\mathcal{B}(\Omega_{T,K})$.

For $T'>T>0$ and $K'>K>0$ we have $\varphi_{T',K'|\Omega_{T,K}} = \varphi_{T,K}$ by uniqueness of the fixed point in $\mathcal{B}(\Omega_{T,K})$, and we can define φ by setting $\varphi_{|\Omega_{T,K}} = \varphi_{T,K}$ for any $T,K>0$. Clearly the function φ thus defined is the unique fixed point of Γ in $\mathcal{B}_{loc}([0,\infty)\times(0,\infty))$.

Since Γ preserves the closed cone of nonnegative functions if f is nonnegative, the fixed point $\varphi_{t_0,K}$ is necessarily nonnegative when f is so. Then by iteration $\varphi_{T,K}\geqslant 0$ for any $T,K>0$, and ultimately $\varphi \geqslant 0$. Similarly, the closed subspace of continuous functions being invariant under Γ when f is continuous, the fixed point φ inherits the continuity of f.

Consider now that f is continuously differentiable on $(0,\infty)$. Unlike the sets of nonnegative or continuous functions, the subspace $\mathcal{C}^1(\Omega_{t_0,K})$ is not closed in $\mathcal{B}(\Omega_{t_0,K})$ for the norm $\|\cdot\|_\infty$. For proving the continuous differentiability of φ we repeat the fixed point argument in the Banach spaces

$$\{g \in \mathcal{C}^1(\Omega_{T,K}),\ g(0,\cdot) = f\}$$

endowed with the norm

$$\|g\|_{\mathcal{C}^1} = \|g\|_\infty + \|\partial_t g\|_\infty + \|x\partial_x g\|_\infty\,.$$

Differentiating Γg with respect to t we get

$$\begin{aligned}
\partial_t(\Gamma g)(t,x) &= \big[x\mathrm{e}^t f'(x\mathrm{e}^t) - B(x\mathrm{e}^t) f(x\mathrm{e}^t)\big]\,\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s} \\
&\qquad\qquad + 2B(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s} g\Big(0,\frac{x\mathrm{e}^t}{2}\Big) \\
&\qquad\qquad\quad + 2\int_0^t B(x\mathrm{e}^\tau)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s}\partial_t g\Big(t-\tau,\frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau \\
&= \mathcal{A}f(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s} + 2\int_0^t B(x\mathrm{e}^\tau)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s}\partial_t g\Big(t-\tau,\frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau
\end{aligned} \tag{11}$$

and differentiating the alternative formulation

$$\Gamma g(t,x) = f(x\mathrm{e}^t)\mathrm{e}^{-\int_x^{x\mathrm{e}^t} B(z)\frac{\mathrm{d}z}{z}} + 2\int_x^{x\mathrm{e}^t} B(y)\mathrm{e}^{-\int_x^y B(z)\frac{\mathrm{d}z}{z}} g\Big(t-\log\big(\frac{y}{x}\big),\frac{y}{2}\Big)\frac{\mathrm{d}y}{y}$$

with respect to x we obtain

$$\begin{aligned}
x\partial_x(\Gamma g)(t,x) &= \Big[\mathcal{A}f(x\mathrm{e}^t) + B(x)f(x\mathrm{e}^t)\Big]\mathrm{e}^{-\int_x^{x\mathrm{e}^t} B(z)\frac{\mathrm{d}z}{z}} - 2B(x)g\Big(t,\frac{x}{2}\Big) \\
&\qquad + 2B(x)\int_x^{x\mathrm{e}^t} B(y)\mathrm{e}^{-\int_x^y B(z)\frac{\mathrm{d}z}{z}} g\Big(t-\log\big(\frac{y}{x}\big),\frac{y}{2}\Big)\frac{\mathrm{d}y}{y} \\
&\qquad\quad + 2\int_x^{x\mathrm{e}^t} B(y)\mathrm{e}^{-\int_x^y B(z)\frac{\mathrm{d}z}{z}} \partial_t g\Big(t-\log\big(\frac{y}{x}\big),\frac{y}{2}\Big)\frac{\mathrm{d}y}{y} \\
&= \Big[\mathcal{A}f(x\mathrm{e}^t) + B(x)f(x\mathrm{e}^t) - 2B(x)f\Big(\frac{x}{2}\Big)\Big]\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\,\mathrm{d}s} \\
&\qquad + 2\int_0^t \big(B(x\mathrm{e}^\tau) - \mathrm{e}^\tau B(x)\big)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s}\partial_t g\Big(t-\tau,\frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau \\
&\qquad + B(x)\int_0^t \mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s}\partial_x g\Big(t-\tau,\frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau.
\end{aligned}$$

On the one hand, using the second expression of $x\partial_x(\Gamma g)(t,x)$ above we deduce that for $g_1,g_2\in\mathcal{C}^1(\Omega_{t_0,K})$ such that $g_1(0,\cdot)=g_2(0,\cdot)=f$ we have

$$\begin{aligned}
&\|\Gamma g_1 - \Gamma g_2\|_{\mathcal{C}^1} \\
&\leqslant t_0 \sup_{(0,K)} B\,\Big(2\,\|g_1-g_2\|_\infty + 2(2+\mathrm{e}^{t_0})\,\|\partial_t g_1 - \partial_t g_2\|_\infty + \|\partial_x g_1 - \partial_x g_2\|_\infty\Big) \\
&\leqslant 2t_0(2+\mathrm{e}^{t_0})\,\sup_{(0,K)} B\;\|g_1-g_2\|_{\mathcal{C}^1}.
\end{aligned}$$

Thus Γ is a contraction for t_0 small enough and this guarantees that the fixed point φ necessarily belongs to $\mathcal{C}^1([0,\infty)\times(0,\infty))$. On the other hand, using the first expression of $x\partial_x(\Gamma g)(t,x)$ we have

$$\partial_t(\Gamma g)(t,x) - x\partial_x(\Gamma g)(t,x) = B(x)\bigg(2g\Big(t,\frac{x}{2}\Big) - \Gamma g(t,x)\bigg)$$

and accordingly the fixed point satisfies $\partial_t\varphi = \mathcal{A}\varphi$. $\square$

From now on, we assume that the division rate B satisfies (10). From Proposition 2 we deduce that Equation (2) generates a positive semigroup on $\dot{\mathcal{B}}$ by setting for any $t \geqslant 0$ and $f \in \mathcal{B}_{loc}(0,\infty)$

$$M_t f := \varphi(t,\cdot).$$

Corollary 3. *The family $(M_t)_{t\geqslant 0}$ defines a semigroup of positive operators on $\mathcal{B}_{loc}(0,\infty)$. If $f \in \mathcal{B}_{loc} \cap \mathcal{C}^1(0,\infty)$ then the function $(t,x) \mapsto M_t f(x)$ is continuously differentiable on $(0,\infty)\times(0,\infty)$ and satisfies*

$$\partial_t M_t f(x) = \mathcal{A} M_t f(x) = M_t \mathcal{A} f(x).$$

Moreover the subspaces $\dot{\mathcal{B}}$ and $\dot{\mathcal{C}}$ are invariant under M_t, and for any $f \in \dot{\mathcal{B}}$ and any $t \geqslant 0$

$$\|M_t f\|_{\dot{\mathcal{B}}} \leqslant \mathrm{e}^t \|f\|_{\dot{\mathcal{B}}}.$$

Proof. The semigroup property $M_{t+s} = M_t M_s$ follows from the uniqueness of the fixed point in the proof of Proposition 2, $(t,x) \mapsto M_{t+s} f(x)$ and $(t,x) \mapsto M_t(M_s f)(x)$ being both solutions with initial distribution $M_s f \in \mathcal{B}_{loc}(0,\infty)$.

The positivity of M_t is given by Proposition 2.

Proposition 2 also provides the regularity of $(t,x) \mapsto M_t f(x)$ when $f \in \mathcal{B}_{loc} \cap \mathcal{C}^1(0,\infty)$, as well as the identity $\partial_t M_t f = \mathcal{A} M_t f$. Besides, if $f \in \mathcal{B}_{loc} \cap \mathcal{C}^1(0,\infty)$ then $\mathcal{A} f \in \mathcal{B}_{loc} \cap \mathcal{C}^1(0,\infty)$ and (11) with $g(t,x) = M_t f(x)$ ensures, still by uniqueness of the fixed point, that $\partial_t M_t f = M_t \mathcal{A} f$.

Simple calculations provide that if $f(x) = x$ then $M_t f(x) = x\mathrm{e}^t$. Together with the positivity of M_t it guarantees that $\|M_t f\|_{\dot{\mathcal{B}}} \leqslant \mathrm{e}^t \|f\|_{\dot{\mathcal{B}}}$ for any f in $\dot{\mathcal{B}}$. In particular $\dot{\mathcal{B}}$ is invariant under M_t, and $\dot{\mathcal{C}}$ also by virtue of Proposition 2. □

We give now another useful property of the positive operators M_t, namely that they preserve increasing pointwise limits.

Lemma 4. *Let $f \in \mathcal{B}_{loc}(0,\infty)$ and let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{B}_{loc}(0,\infty)$ be an increasing sequence that converges pointwise to f, i.e for all $x > 0$*

$$f(x) = \lim_{n\to\infty} \uparrow f_n(x).$$

Then for all $t \geqslant 0$ and all $x > 0$

$$M_t f(x) = \lim_{n\to\infty} M_t f_n(x).$$

Proof. Let f and $(f_n)_{n\in\mathbb{N}}$ satisfy the assumptions of the lemma. For all $t \geqslant 0$, the positivity of M_t ensures that the sequence $(M_t f_n)_{n\in\mathbb{N}}$ is increasing and bounded by $M_t f$. Denote by $g(t,x)$ the limit of $M_t f_n(x)$. Using the monotone convergence theorem, we get by passing to the limit in

$$M_t f_n(x) = f_n(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s} + 2\int_0^t B(x\mathrm{e}^\tau)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s} M_{t-\tau} f_n\Big(\frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau$$

that

$$g(t,x)=f(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s}+2\int_0^t B(x\mathrm{e}^\tau)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s}g\Big(t-\tau,\frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau.$$

By uniqueness property we deduce that $g(t,x)=M_tf(x)$. $\square$

3.2 Construction of a measure solution

Using the results in Section 3.1, we define a left action of the semigroup $(M_t)_{t\geqslant 0}$ on $\dot{\mathcal{M}}$. To do so we first set for $t\geqslant 0$, $\mu\in\dot{\mathcal{M}}_+$, and $A\subset(0,\infty)$ Borel set

$$(\mu M_t)(A):=\int_0^\infty M_t\mathbb{1}_A\,\mathrm{d}\mu$$

and verify that μM_t such defined is a positive measure on $(0,\infty)$.

Lemma 5. *For all $\mu\in\dot{\mathcal{M}}_+$ and all $t\geqslant 0$, μM_t defines a positive measure. Additionally $\mu M_t\in\dot{\mathcal{M}}_+$ and for any $f\in\dot{\mathcal{B}}$*

$$(\mu M_t)(f)=\mu(M_tf).$$

Proof. Let $\mu\in\dot{\mathcal{M}}_+$ and $t\geqslant 0$. We first check that μM_t is a positive measure.

Clearly $\mu M_t(A)\geqslant 0$ for any Borel set A, and $\mu M_t(\varnothing)=\int_0^\infty M_t\mathbf{0}\,\mathrm{d}\mu=0$.

Let $(A_n)_{n\in\mathbb{N}}$ be a countable sequence of disjoint Borel sets of $(0,\infty)$ and define $f_n=\sum_{k=0}^n\mathbb{1}_{A_k}=\mathbb{1}_{\bigsqcup_{k=0}^n A_k}$. For every integer n, one has

$$\mu M_t\left(\bigsqcup_{k=0}^n A_k\right)=\int_0^\infty M_tf_n\,\mathrm{d}\mu=\sum_{k=0}^n\int_0^\infty M_t(\mathbb{1}_{A_k})\mathrm{d}\mu=\sum_{k=0}^n\mu M_t(A_k).$$

The sequence $(f_n)_{n\in\mathbb{N}}$ is increasing and its pointwise limit is $f=\mathbb{1}_{\bigsqcup_{k=0}^\infty A_k}$, which belongs to $\mathcal{B}_{loc}(0,\infty)$. We deduce from Lemma 4 and the monotone convergence theorem that

$$\lim_{n\to\infty}\mu M_t\left(\bigsqcup_{k=0}^n A_k\right)=\lim_{n\to\infty}\int_0^\infty M_tf_n\,\mathrm{d}\mu=\int_0^\infty M_tf\,\mathrm{d}\mu=\mu M_t\left(\bigsqcup_{k=0}^\infty A_k\right)$$

where the limit lies in $[0,+\infty]$. This ensures that

$$\mu M_t\left(\bigsqcup_{k=0}^\infty A_k\right)=\sum_{k=0}^\infty\mu M_t(A_k)$$

and μM_t thus satisfies the definition of a positive measure.

By definition of μM_t, the identity $(\mu M_t)(f)=\mu(M_tf)$ is clearly true for any simple function f. Since any nonnegative measurable function is the increasing pointwise limit of simple functions, Lemma 4 ensures that it is also valid in $[0,+\infty]$ for any nonnegative $f\in\mathcal{B}_{loc}(0,\infty)$. Considering $f(x)=x$ we get $(\mu M_t)(f)=\mu(f)\mathrm{e}^t<+\infty$, so that $\mu M_t\in\dot{\mathcal{M}}_+$. Finally, decomposing $f\in\dot{\mathcal{B}}$ as $f=f_+-f_-$ we readily obtain that $(\mu M_t)(f)=\mu(M_tf)$.

$\square$

Now for $\mu \in \dot{\mathcal{M}}$ and $t \geqslant 0$, we naturally define $\mu M_t \in \dot{\mathcal{M}}$ by

$$\mu M_t = \mu_+ M_t - \mu_- M_t.$$

It is then clear that the identity $(\mu M_t)(f) = \mu(M_t f)$ is still valid for $\mu \in \dot{\mathcal{M}}$ and $f \in \dot{\mathcal{B}}$.

Proposition 6. *The left action of $(M_t)_{t\geqslant 0}$ defines a positive semigroup in $\dot{\mathcal{M}}$, which satisfies for all $t \geqslant 0$ and all $\mu \in \dot{\mathcal{M}}$*

$$\|\mu M_t\|_{\dot{\mathcal{M}}} \leqslant \mathrm{e}^t \|\mu\|_{\dot{\mathcal{M}}}.$$

Proof. Using the duality relation $(\mu M_t)(f) = \mu(M_t f)$, it is a direct consequence of Corollary 3. □

Finally we prove that the (left) semigroup $(M_t)_{t\geqslant 0}$ yields the unique measure solutions to Equation (1).

Theorem 7. *For any $\mu \in \dot{\mathcal{M}}$, the family $(\mu M_t)_{t\geqslant 0}$ is the unique solution to Equation* (1)*, in the sense of Definition 1, with initial distribution μ.*

Proof. Let $\mu \in \dot{\mathcal{M}}$. We first check that $t \mapsto (\mu M_t)(f)$ is continuous for any $f \in \dot{\mathcal{C}}$ by writing

$$\begin{aligned}
|(\mu M_t)(f) - \mu(f)| \leqslant & \left| \int_0^\infty f(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s} - f(x)\,\mu(\mathrm{d}x) \right| \\
& + \left| 2\int_0^\infty \int_0^t B(x\mathrm{e}^\tau)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s} M_{t-\tau} f\Big(\frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau\,\mu(\mathrm{d}x) \right| \\
\leqslant & \int_0^\infty \left| f(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s} - f(x) \right| |\mu|(\mathrm{d}x) \\
& + \mathrm{e}^t \|f\|_{\dot{\mathcal{B}}} \int_0^\infty (1 - \mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s}) x\,|\mu|(\mathrm{d}x).
\end{aligned}$$

The two terms in the right hand side vanish as t tends to 0 by dominated convergence theorem and the continuity of $t \mapsto (\mu M_t)(f)$ follows from the semigroup property.

Now consider $f \in D(\mathcal{A})$. Integrating $\partial_t M_t f = M_t \mathcal{A} f$ between 0 and t we obtain for all $x > 0$

$$M_t f(x) = f(x) + \int_0^t M_s(\mathcal{A}f)(x)\,\mathrm{d}s.$$

By definition of $D(\mathcal{A})$, the function $\mathcal{A}f$ belongs to $\dot{\mathcal{B}}$, so we deduce the inequality $|M_s(\mathcal{A}f)(x)| \leqslant \|\mathcal{A}f\|\mathrm{e}^s x$ and we can use Fubini's theorem to get by integration against μ

$$\mu(M_t f) = \mu(f) + \mu\left(\int_0^t M_s(\mathcal{A}f)\mathrm{d}s\right) = \mu(f) + \int_0^t \mu(M_s(\mathcal{A}f))\mathrm{d}s.$$

The duality relation $(\mu M_t)(f) = \mu(M_t f)$ then guarantees that (μM_t) satisfies (5).

It remains to check the uniqueness. Let $(\mu_t)_{t\geqslant 0}$ be a solution to Equation (1) with $\mu_0 = \mu$. Recall that it implies in particular that $t \mapsto \mu_t(f)$ is continuous for any $f \in \dot{\mathcal{C}}$, and consequently $t \to \mu_t$ is locally bounded for the norm $\|\cdot\|_{\dot{\mathcal{M}}}$ due to the uniform boundedness principle. We want to verify that $\mu_t = \mu M_t$ for all $t \geqslant 0$. Fix $t > 0$ and $f \in \mathcal{C}_c^1(0,\infty)$, and let us compute the derivative of the mapping

$$s \mapsto \int_0^s \mu_\tau(M_{t-s}f)\,\mathrm{d}\tau$$

defined on $[0,t]$. For $0 < s < s+h < t$ we have

$$\begin{aligned}\frac{1}{h}\Big[\int_0^{s+h}\mu_\tau(M_{t-s-h}f)\,\mathrm{d}\tau - \int_0^{s}\mu_\tau(M_{t-s}f)\,\mathrm{d}\tau\Big] &= \frac{1}{h}\int_s^{s+h}\mu_\tau(M_{t-s}f)\,\mathrm{d}\tau\\ + \int_s^{s+h}\mu_\tau\Big(\frac{M_{t-s-h}f - M_{t-s}f}{h}\Big)\,\mathrm{d}\tau &+ \int_0^{s}\mu_\tau\Big(\frac{M_{t-s-h}f - M_{t-s}f}{h}\Big)\,\mathrm{d}\tau.\end{aligned}$$

The convergence of the first term is a consequence of the continuity of $\tau \mapsto \mu_\tau(M_{t-s}f)$

$$\frac{1}{h}\int_s^{s+h}\mu_\tau M_{t-s}f\,\mathrm{d}\tau \xrightarrow[h\to 0]{} \mu_s M_{t-s}f.$$

For the second term we use that

$$M_{t-s}f - M_{t-s-h}f = M_{t-s-h}\int_0^h \partial_\tau M_\tau f\,\mathrm{d}\tau = M_{t-s-h}\int_0^h M_\tau \mathcal{A}f\,\mathrm{d}\tau$$

to get, since $\tau \mapsto \|\mu_\tau\|_{\dot{\mathcal{M}}}$ is locally bounded,

$$\left|\int_s^{s+h}\mu_\tau\Big(\frac{M_{t-s-h}f - M_{t-s}f}{h}\Big)\,\mathrm{d}\tau\right| \leqslant h \sup_{\tau\in[0,t]}\|\mu_\tau\|_{\dot{\mathcal{M}}}\ \|\mathcal{A}f\|_{\dot{\mathcal{B}}}\,\mathrm{e}^{t-s} \xrightarrow[h\to 0]{} 0.$$

For the last term we have, by dominated convergence and using the identity $\partial_t M_t f = \mathcal{A}M_t f$,

$$\int_0^s \mu_\tau \frac{M_{t-s-h}f - M_{t-s}f}{h}\,\mathrm{d}\tau \xrightarrow[h\to 0]{} -\int_0^s \mu_\tau \mathcal{A}M_{t-s}f\,\mathrm{d}\tau.$$

Finally we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \mu_\tau(M_{t-s}f)\,\mathrm{d}\tau = \mu_s(M_{t-s}f) - \int_0^s \mu_\tau(\mathcal{A}M_{t-s}f)\,\mathrm{d}\tau = \mu_0(M_{t-s}f).$$

To obtain the last equality, one has to notice that $f \in D(\mathcal{A})$, so Corollary 3 ensures that $M_{t-s}f \in D(\mathcal{A})$ can be used in Definition 1 in place of f. Integrating between $s = 0$ and $s = t$ we obtain, since $\mu_0 = \mu$,

$$\int_0^t \mu_\tau(f)\,\mathrm{d}\tau = \int_0^t \mu(M_{t-s}f)\,\mathrm{d}s = \int_0^t (\mu M_\tau)(f)\,\mathrm{d}\tau$$

then by differentiation with respect to t

$$\mu_t(f) = (\mu M_t)(f).$$

By density of $\mathcal{C}^1_c(0,\infty)$ in $\dot{\mathcal{C}}_0$, it ensures that $\mu_t = \mu M_t$. $\square$

4 Long time asymptotics

To study the long time behavior of the measure solutions to Equation (1) we proceed by duality by first analyzing Equation (2). The method relies on the general relative entropy structure of this dual problem. From now on, we assume that the division rate B satisfies (8), so that the Perron eigenelements exist.

Lemma 8 (General Relative Entropy)**.** *Let $H : \mathbb{R} \to \mathbb{R}$ be a differentiable convex function. Then for all $f \in \dot{\mathcal{B}} \cap \mathcal{C}^1(0,\infty)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty x\,\mathcal{U}(x)H\left(\frac{M_tf(x)}{x\,\mathrm{e}^t}\right)\mathrm{d}x = -D^H[\mathrm{e}^{-t}M_tf] \leqslant 0$$

with D^H defined on $\dot{\mathcal{B}}$ by

$$D^H[f] = \int_0^\infty xB(x)\mathcal{U}(x) \\ \left[H'\left(\frac{f(x)}{x}\right)\left(\frac{f(x)}{x} - \frac{f(x/2)}{x/2}\right) + H\left(\frac{f(x/2)}{x/2}\right) - H\left(\frac{f(x)}{x}\right)\right]\mathrm{d}x.$$

Proof. For $f \in \dot{\mathcal{B}} \cap \mathcal{C}^1(0,\infty)$ the function $(t,x) \mapsto M_tf(x)$ is continuously differentiable and verifies $\partial_t M_tf(x) = \mathcal{A}M_tf(x)$, see Corollary 3. Simple computations then yield, using that $\mathcal{U}$ satisfies (6),

$$\begin{aligned}\left(\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\right)&\left(x\,\mathcal{U}(x)H\left(\frac{M_tf(x)}{x\,\mathrm{e}^t}\right)\right) \\ &= x\,\mathcal{U}(x)B(x)H'\left(\frac{M_tf(x)}{x\,\mathrm{e}^t}\right)\left(\frac{M_tf(x/2)}{x/2} - \frac{M_tf(x)}{x}\right) \\ &\quad - H\left(\frac{M_tf(x)}{x\,\mathrm{e}^t}\right)x\left(4B(2x)\mathcal{U}(2x) - B(x)\mathcal{U}(x) - \mathcal{U}(x)\right)\end{aligned}$$

and the conclusion follows by integration. $\square$

This result reveals the lack of coercivity of the equation in the sense that the dissipation $D^H[f]$ does not vanish only for $f(x) = \phi(x) = x$ but for any function f such that $f(2x) = 2f(x)$ for all $x > 0$. In particular all the eigenfunctions ϕ_k satisfy this relation, so $D^H[\mathrm{Re}(\phi_k)] = D^H[\mathrm{Im}(\phi_k)] = 0$. More precisely we have the following result about the space

$$X := \left\{f \in \dot{\mathcal{C}}^{\mathbb{C}} \mid \forall x > 0,\ f(2x) = 2f(x)\right\}.$$

Lemma 9. *We have the identity*

$$X = \overline{\operatorname{span}}(\phi_k)_{k\in\mathbb{Z}}$$

and more specifically any $f \in X$ is the limit in $(\dot{\mathcal{C}}^{\mathbb{C}}, \|\cdot\|)$ of a Fejér type sum

$$f = \lim_{N\to\infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \nu_k(f)\phi_k.$$

Proof. The vector subspace X contains all the ϕ_k and is closed in $(\dot{\mathcal{C}}^{\mathbb{C}}, \|\cdot\|)$, so it contains $\overline{\operatorname{span}}(\phi_k)_{k\in\mathbb{Z}}$.

To obtain the converse inclusion, we consider $f \in X$ and we write it as

$$f(x) = x\,\theta(\log x)$$

with $\theta : \mathbb{R} \to \mathbb{C}$ a continuous $\log 2$-periodic function. The Fejér theorem ensures that the Fejér sum, namely the Cesàro means of the Fourier series

$$\sigma_N(\theta)(y) := \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} \hat{\theta}(k)\mathrm{e}^{\frac{2ik\pi}{\log 2}y} = \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N}\right)\hat{\theta}(k)\mathrm{e}^{\frac{2ik\pi}{\log 2}y}$$

where

$$\hat{\theta}(k) = \frac{1}{\log 2}\int_0^{\log 2} \theta(y)\mathrm{e}^{-\frac{2ik\pi}{\log 2}y}\mathrm{d}y$$

converges uniformly on $\mathbb{R}$ to θ. We deduce that the sequence $(F_N(f))_{N\geqslant 1} \subset \operatorname{span}(\phi_k)_{k\in\mathbb{Z}}$ defined by

$$F_N(f)(x) := x\,\sigma_N(\theta)(\log x) = \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N}\right)\hat{\theta}(k)\phi_k(x)$$

converges to f in norm $\|\cdot\|_{\dot{\mathcal{B}}}$.

To conclude it remains to verify that $\hat{\theta}(k) = \nu_k(f)$. Since $\int_0^\infty x\,\mathcal{U}(x)\mathrm{d}x = 1$ by definition and $\lambda_k \neq \lambda_l$ when $k \neq l$, we have that $\nu_k(\phi_l) = \delta_{kl}$, the Kronecker delta function. We deduce that for any positive integer N

$$\nu_k(F_N(f)) = \begin{cases} 0 \text{ if } N < |k|, \\ (1 - \frac{|k|}{N})\hat{\theta}(k) \text{ otherwise.} \end{cases}$$

As a consequence for all $N \geqslant |k|$ we have

$$|\nu_k(f) - \hat{\theta}(k)| \leqslant \|f - F_N(f)\|_{\dot{\mathcal{B}}} + \frac{|k|}{N}\,\|f\|_{\dot{\mathcal{B}}}$$

and this gives the desired identity by letting N tend to infinity.

∎

We have shown in the proof of Lemma 9 that the Fejér sums F_N can be extended to $\dot{\mathcal{C}}^{\mathbb{C}}$ by setting

$$F_N(f) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \nu_k(f)\phi_k.$$

The limit when $N \to \infty$, provided it exists, is a good candidate for defining a relevant projection on X. Using Lemma 8 we prove in the following theorem that the sequence $(F_N(f))_{n\geqslant 1}$ converges in X for any $f \in \mathcal{C}_c^1(0,\infty)$, and that the limit extends into a linear operator $\dot{\mathcal{C}}_0 \to X$ which provides the asymptotic behavior of $(M_t)_{t\geqslant 0}$ on $\mathcal{C}_0$.

Theorem 10. *For any $f \in \mathcal{C}_c^1(0,\infty)$ and any $t \geqslant 0$ the sequence*

$$F_N(\mathrm{e}^{-t}M_t f) = \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N}\Big)\nu_k(f)\mathrm{e}^{\frac{2i\pi k}{\log 2}t}\phi_k$$

converges in $\dot{\mathcal{C}}$ and the limit $R_t f$ defines a $\log 2$-periodic family of bounded linear operators $R_t : \dot{\mathcal{C}}_0 \to X \cap \dot{\mathcal{C}}$. Moreover for all $f \in \dot{\mathcal{C}}_0$

$$\mathrm{e}^{-t}M_t f - R_t f \xrightarrow[t\to\infty]{} 0$$

locally uniformly on $(0,\infty)$.

Notice that R_0 is actually a projector from $\dot{\mathcal{C}}_0 \oplus X$ onto X.

Proof. We know from Corollary 3 that $\mathrm{e}^{-t}M_t$ is a contraction for $\|\cdot\|$. Let $f \in \mathcal{C}_c^1(0,\infty)$. We have $\mathcal{A}f \in \dot{\mathcal{C}}$ and so $\partial_t(\mathrm{e}^{-t}M_t f) = M_t(\mathcal{A}f - f)$ is bounded in time in $\dot{\mathcal{C}}$. Since $x\partial_x M_t f(x) = \partial_t M_t f(x) - B(x)(2M_t f(x/2) - M_t f(x))$ and B is locally bounded we deduce that $\mathrm{e}^{-t}\partial_x M_t f$ is locally bounded on $(0,\infty)$ uniformly in $t \geqslant 0$. So the Arzela-Ascoli theorem ensures that there exists a subsequence of $(\mathrm{e}^{-t-n\log 2}M_{t+n\log 2}f(x))_{n\geqslant 0}$ which converges locally uniformly on $[0,\infty) \times (0,\infty)$ to a limit $h(t,x)$, with $h(t,\cdot) \in \dot{\mathcal{C}}$ for all $t \geqslant 0$. We now use Lemma 8 to identify this limit. The dissipation of entropy for the convex function $H(x) = x^2$, denoted D^2, reads

$$D^2[f] = \int_0^\infty xB(x)\mathcal{U}(x) \left| \frac{f(x/2)}{x/2} - \frac{f(x)}{x} \right|^2 \mathrm{d}x.$$

The general relative entropy inequality in Lemma 8 guarantees that

$$\int_0^\infty D^2[\mathrm{e}^{-t}M_t f]\mathrm{d}t < +\infty$$

and as a consequence, for all $T > 0$,

$$\int_0^T D^2[\mathrm{e}^{-t-n\log 2}M_{t+n\log 2}f]\mathrm{d}t = \int_{n\log 2}^{T+n\log 2} D^2[\mathrm{e}^{-t}M_t f]\mathrm{d}t \xrightarrow[n\to\infty]{} 0.$$

From the Cauchy-Schwarz inequality we deduce that

$$\frac{\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f(x/2)}{x/2} - \frac{\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f(x)}{x} \to 0$$

in the distributional sense on $(0,\infty)^2$, and since $\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f(x)$ converges locally uniformly to $h(t,x)$ we get that for all $t \geqslant 0$ and $x > 0$

$$\frac{h(t,x/2)}{x/2} - \frac{h(t,x)}{x} = 0.$$

This means that $h(t,\cdot) \in X$ for all $t \geqslant 0$, and Lemma 9 then ensures that

$$h(t,\cdot) = \lim_{N\to\infty} \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N}\Big) \nu_k(h(t,\cdot))\phi_k.$$

Since by definition of $\mathcal{U}_k$ we have $\nu_k M_t = \mathrm{e}^{\lambda_k t} \nu_k$, the dominated convergence theorem yields

$$\nu_k(h(t,\cdot)) = \lim_{n\to\infty} \mathrm{e}^{-t-n\log 2} (\nu_k M_{t+n\log 2}) f = \mathrm{e}^{\frac{2ik\pi}{\log 2} t} \nu_k(f)$$

and so

$$h(t,\cdot) = \lim_{N\to\infty} \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N}\Big) \nu_k(f) \mathrm{e}^{\frac{2ik\pi}{\log 2} t} \phi_k = \lim_{N\to\infty} F_N(\mathrm{e}^{-t} M_t f).$$

This guarantees that $(F_N(M_t f))_{N\geqslant 1}$ is convergent in $\dot{\mathcal{C}}$. Its limit denoted by $R_t f$ clearly defines a linear operator $R_t : \mathcal{C}_c^1(0,\infty) \to X \cap \dot{\mathcal{C}}$. Moreover by local uniform convergence of $\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f$ to $R_t f$ we get that

$$\|R_t f\|_{\dot{\mathcal{B}}} \leqslant \limsup_{n\to\infty} \big\|\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f\big\|_{\dot{\mathcal{B}}} \leqslant \|f\|_{\dot{\mathcal{B}}}.$$

Thus R_t is bounded and it extends uniquely to a contraction $\dot{\mathcal{C}}_0 \to X \cap \dot{\mathcal{C}}$. The local uniform convergence of $\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f(x)$ to $R_t f(x)$ for $f \in \mathcal{C}_c^1(0,\infty)$ also guarantees the local uniform convergence of $\mathrm{e}^{-t} M_t f - R_t f$ to zero when $t \to +\infty$. Indeed, letting K be a compact set of $(0,\infty)$ and defining for all $t \geqslant 0$ the integer part $n := \big\lfloor \frac{t}{\log 2} \big\rfloor$, so that $t' := t - k\log 2 \in [0,\log 2]$, one has

$$\begin{aligned}
&\sup_{x\in K} |\mathrm{e}^{-t} M_t f(x) - R_t f(x)| \\
&= \sup_{x\in K} |\mathrm{e}^{-(n\log 2 + t')} M_{n\log 2 + t'} f(x) - R_{t'} f(x)| \\
&\leqslant \sup_{x\in K} \sup_{s\in[0,\log 2]} |\mathrm{e}^{-(n\log 2 + s)} M_{n\log 2 + s} f(x) - R_s f(x)|.
\end{aligned}$$

This convergence extends to any $f \in \dot{\mathcal{C}}_0$ by density. $\square$

Due to the Riesz representation $\dot{\mathcal{M}} \simeq \dot{\mathcal{C}}_0'$, we can define a $\log 2$-periodic contraction semigroup R_t on $\dot{\mathcal{M}}$ by setting for all $\mu \in \dot{\mathcal{M}}$ and all $f \in \dot{\mathcal{C}}_0$

$$(\mu R_t)(f) := \mu(R_t f).$$

Theorem 10 then yields the weak-* convergence result in Theorem 1 since $\rho_t = \mu_0 R_t$. The following theorem readily implies the uniform exponential convergence in weighted total variation norm.

Theorem 11. *Let $r_1, r_2 \in \mathbb{R}$ such that $r_1 < 1 < r_2$ and define $w(x) = x^{r_1} + x^{r_2}$. Then R_t is a bounded endomorphism of $\mathcal{C}(w)$ for any $t \geqslant 0$, and there exist explicit constants $C \geqslant 1$ and $a > 0$ such that for all $f \in \mathcal{B}(w)$ and all $t \geqslant 0$*

$$\left\| \mathrm{e}^{-t} M_t f - R_t f \right\|_{\mathcal{B}(w)} \leqslant C \mathrm{e}^{-at} \left\| f - R_0 f \right\|_{\mathcal{B}(w)}.$$

Proof. Harris's ergodic theorem provides conditions for the uniform ergodicity in weighted supremum norm of Markov operators, namely operators P such that $P\mathbf{1} = \mathbf{1}$. We refer for instance to the recent paper [40] for a concise and direct proof of this theorem. The semigroup $(M_t)_{t\geqslant 0}$ is not a family of Markov operators so we consider the rescaled semigroup $(P_t)_{t\geqslant 0}$ defined on $\mathcal{B}(0,\infty)$ by

$$P_t f(x) := \frac{M_t(\phi f)(x)}{\mathrm{e}^t \phi(x)}.$$

Since $\phi(x) = x$ verifies $M_t \phi = \mathrm{e}^t \phi$, the family $(P_t)_{t\geqslant 0}$ is clearly a semigroup of Markov operators. However, since the long time behavior of $(P_t)_{t\geqslant 0}$ consists in persistent oscillations, there is no hope that Harris's ergodic theorem applies to this semigroup. The idea is to rather apply it to a discrete time semigroup on a discrete state space. Let us fix $x > 0$ until the end of the proof, and define

$$\mathbf{X}_x := \left\{ y \in (0,\infty) : \exists m \in \mathbb{Z}, y = 2^m x \right\}.$$

The left action of $P_{\log 2}$ defines an operator on the measures on $\mathbf{X}_x$. Let us give a rigorous proof of this claim. It is easily seen in the proof of Proposition 2 that if f vanishes on $\mathbf{X}_z$, then Γ leaves invariant the set of functions g such that $g(t, y) = 0$ for all $t \geqslant 0$ and $y \in \mathbf{X}_{\mathrm{e}^{-t}z}$. It implies that the fixed point $M_t f$ belongs to this set, and consequently so does $P_t f$. In other words, if $y \in \mathbf{X}_{\mathrm{e}^{-t}z}$ then $\operatorname{supp}(\delta_y P_t) \subset \mathbf{X}_z$. Applying this to $z = x\mathrm{e}^t$ ensures that if $\operatorname{supp}\mu \subset \mathbf{X}_x$ then $\operatorname{supp}(\mu P_t) \subset \mathbf{X}_{\mathrm{e}^t x}$. Since $\mathbf{X}_{2x} = \mathbf{X}_x$ we deduce that $P_{\log 2}$ leaves invariant the elements of $\dot{\mathcal{M}}$ with support included in $\mathbf{X}_x$.

Let us denote by P the operator $P_{\log 2}$ seen as a Markov operator on the state space $\mathbf{X}_x$. We will prove that P^n satisfies Assumptions 1 and 2 in [40] for some positive integer n and the Lyapunov function $V(x) = x^{q_1} + x^{q_2}$ with $q_1 < 0 < q_2$. To do so we study the continuous time semigroup $(P_t)_{t\geqslant 0}$. Its infinitesimal generator is given by

$$\widetilde{\mathcal{A}} f(x) = x f'(x) + B(x)\big(f(x/2) - f(x)\big)$$

and it satisfies the Duhamel formula

$$P_t f(x) = f(x\mathrm{e}^t)\mathrm{e}^{-\int_0^t B(x\mathrm{e}^s)\mathrm{d}s} + \int_0^t B(x\mathrm{e}^\tau)\mathrm{e}^{-\int_0^\tau B(x\mathrm{e}^s)\mathrm{d}s} P_{t-\tau} f\Big(\frac{x\mathrm{e}^\tau}{2}\Big)\mathrm{d}\tau \quad (12)$$

which is the same as in Proposition 2 but without the factor 2 before the integral. We easily check that

$$\widetilde{\mathcal{A}}V(x) = \big[q_1 + (2^{-q_1} - 1)B(x)\big]x^{q_1} + \big[q_2 + (2^{-q_2} - 1)B(x)\big]x^{q_2}.$$

Since B is continuous, $B(x) \to 0$ at $x = 0$ and $B(x) \to +\infty$ as $x \to +\infty$, we see that for any $\omega \in (0, -q_1)$ the continuous function $\widetilde{\mathcal{A}}V + \omega V$ is bounded from above, or in other words there exists a constant $K > 0$ such that

$$\widetilde{\mathcal{A}}V \leqslant -\omega(V - K).$$

Since $\partial_t P_t V = P_t \tilde{\mathcal{A}} V$ and $\widetilde{\mathcal{A}}K = 0$ we deduce from Grönwall's lemma that

$$P_t V \leqslant \mathrm{e}^{-\omega t} V + K$$

for all $t \geqslant 0$. In particular, since $V \geqslant 1$, this inequality also ensures that $P_t V \leqslant (\mathrm{e}^{-\omega t} + K)V$ and consequently $\mathcal{B}(V)$ is invariant under P_t. In terms of the original semigroup $(M_t)_{t\geqslant 0}$ this yields that

$$\left\|\mathrm{e}^{-t} M_t f\right\|_{\mathcal{B}(w)} \leqslant (\mathrm{e}^{-\omega t} + K)\left\|f\right\|_{\mathcal{B}(w)} \quad (13)$$

for all $t \geqslant 0$ and $f \in \mathcal{B}(w)$ with $w(x) = xV(x) = x^{1+q_1} + x^{1+q_2}$. As a by-product, it guarantees that R_t is a bounded endomorphism of $\mathcal{C}(w)$ since

$$\left\|R_t f\right\|_{\mathcal{B}(w)} \leqslant \limsup_{n\to\infty} \left\|\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f\right\|_{\mathcal{B}(w)} \leqslant K \left\|f\right\|_{\mathcal{B}(w)}.$$

We also deduce that for all integer $n \geqslant 1$ and all $y \in \mathbf{X}_x$

$$P^n V(y) = P_{n\log 2} V(y) \leqslant \gamma V(y) + K$$

with $\gamma = \mathrm{e}^{-\omega\log 2} \in (0,1)$. Assumption 1 of [40] is thus satisfied by P^n for any integer $n \geqslant 1$, with constants which do not depend on n. We will now prove that Assumption 2 is verified for some $n \geqslant 1$ on the sub-level set $\mathcal{S} = \{y \in \mathbf{X}_x : V(y) \leqslant R\}$ for some $R > 2K/(1-\gamma)$.

Fix $R > 2K/(1-\gamma)$ and let $\xi_1, \xi_2 \in \mathbf{X}_x$ be such that $\mathcal{S} \subset [\xi_1, \xi_2]$ and $\xi_2 > b_1$, where b_1 is defined in (8). Define $\overline{\mathcal{S}} := [\xi_1, \xi_2] \cap \mathbf{X}_x \supset \mathcal{S}$ and let us index this set by $\xi_1 = x_0 < x_1 < \cdots < x_{n_0} = \xi_2$, meaning that $\overline{\mathcal{S}} = \{x_0, \cdots, x_{n_0}\}$. We prove by induction on n that for all $n \in \{0, \cdots, n_0\}$, there exists $c_n > 0$ such that for all $f : \mathbf{X}_x \to [0, \infty)$ and all $y \in \overline{\mathcal{S}}$

$$P^n f(y) \geqslant c_n f(\min(2^n y, x_{n_0})). \quad (14)$$

It is trivially satisfied for $n = 0$ with $c_0 = 1$. Assume now that (14) is verified for some $n \in \{0, \cdots, n_0 - 1\}$. Iterating once the Duhamel formula (12) and taking $t = \log 2$ we get that for all $f : \mathbf{X}_x \to [0, \infty)$ and all $y \in \overline{\mathcal{S}}$

$$Pf(y) \geqslant \eta f(2y) + \eta^2 \bigg(\int_0^{\log 2} B(y\mathrm{e}^\tau)\mathrm{d}\tau \bigg) f(y)$$

with $\eta := \exp\big(-\int_{\xi_1}^{2\xi_2} B \big) > 0$. Applying this inequality to $P^n f$ instead of f yields that for all $f : \mathbf{X}_x \to [0, \infty)$ and all $y \in \overline{\mathcal{S}}$

$$P^{n+1} f(y) \geqslant \eta P^n f(2y) + \eta^2 \bigg(\int_0^{\log 2} B(y\mathrm{e}^\tau)\mathrm{d}\tau \bigg) P^n f(y).$$

If $y < x_{n_0}$ then we get by induction hypothesis (14) that

$$P^{n+1} f(y) \geqslant \eta P_n f(2y) \geqslant c_n \eta f(\min(2^{n+1} y, x_{n_0})).$$

If $y = x_{n_0}$ we use that $x_{n_0} = \xi_2 > b_1$ to get

$$P^{n+1} f(x_{n_0}) \geqslant \eta^2 \bigg(\int_0^{\log 2} B(\xi_2\, \mathrm{e}^\tau)\mathrm{d}\tau \bigg) P^n f(x_{n_0}) \geqslant \eta^2 K_1 \xi_2^{\gamma_1} \frac{2^{\gamma_1} - 1}{\gamma_1} P^n f(x_{n_0}).$$

We can thus take $c_{n+1} = \min(c_n \eta, \eta^2 K_1 \xi_2^{\gamma_1} (2^{\gamma_1} - 1)/\gamma_1) > 0$. Now that (14) is proved for all $n \in \{0, \cdots, n_0\}$ we take $n = n_0$ and obtain

$$\inf_{y \in \mathcal{S}} \delta_y P^{n_0} \geqslant c_{n_0} \delta_{x_{n_0}}$$

which is Assumption 2 in [40] with $\alpha = c_{n_0}$ and $\nu = \delta_{x_{n_0}}$.

We are in position to apply the Harris's ergodic theorem. We get the existence of an invariant measure μ^x on $\mathbf{X}_x$, which integrate V, and constants $C \geqslant 1$ and $\varrho \in (0, 1)$ such that for all $f \in \mathcal{B}(\mathbf{X}_x, V)$ and all $m \in \mathbb{N}$

$$\sup_{y \in \mathbf{X}_x} \frac{|P^{mn_0} f(y) - \mu^x(f)|}{V(y)} \leqslant C\varrho^m \sup_{y \in \mathbf{X}_x} \frac{|f(y) - \mu^x(f)|}{V(y)}.$$

Notice that these constants are independent of x, since in [40, Theorem 1.3] they are given explicitly in terms of the constants α, γ and K, that do not depend on x in our calculations above. In particular it implies that $P^{mn_0} f(y)$ converges to $\mu^x(f)$ as $m \to \infty$ for all $y \in \mathbf{X}_x$. But, defining $t_0 = n_0 \log 2$, we know from Theorem 10 that $P_{mt_0} f(y) \to R_0(\phi f)(y)/y$ for all $f \in \mathcal{B}(V)$ as $m \to \infty$. So we obtain, taking $y = x$ in the left hand side, that for all $f \in \mathcal{B}(V)$ and all $m \in \mathbb{N}$

$$\frac{|P_{mt_0} f(x) - R_0(\phi f)(x)/x|}{V(x)} \leqslant C\varrho^m \sup_{y \in \mathbf{X}_x} \frac{|f(y) - R_0(\phi f)(y)/y|}{V(y)}.$$

Still using the function $w(x) = xV(x)$, this yields in terms of $(M_t)_{t \geqslant 0}$ that for all $f \in \mathcal{B}(w)$ and all $m \in \mathbb{N}$

$$\frac{|\mathrm{e}^{-mt_0} M_{mt_0} f(x) - R_0 f(x)|}{w(x)} \leqslant C\varrho^m \sup_{y \in (0,\infty)} \frac{|f(y) - R_0 f(y)|}{w(y)}.$$

Since we chose any $x \in (0,\infty)$ and the constants C and ϱ are independent of x, we finally proved that for all $f \in \mathcal{B}(w)$ and all $m \in \mathbb{N}$

$$\left\| \mathrm{e}^{-mt_0} M_{mt_0} f - R_0 f \right\|_{\mathcal{B}(w)} \leqslant C\varrho^m \left\| f - R_0 f \right\|_{\mathcal{B}(w)}.$$

As $R_{mt_0} = R_0$ by periodicity, this gives the result of Theorem 11 for discrete times $t = mt_0$. It easily extends to continuous times due to the bound (13).

□

We finish by giving consequences of Theorems 10 and 11 in terms of mean ergodicity. Since the limit is $\log 2$-periodic we expect by taking the mean in time of the semigroup to get alignment on the Perron eigenfunction. The results are given in the following corollary for the right semigroup, but again they can readily be transposed to the left action on measures by duality.

Corollary 12. *For any $f \in \dot{\mathcal{C}}_0$ the two mappings*

$$t \mapsto \frac{1}{\log 2}\int_t^{t+\log 2} \mathrm{e}^{-s} M_s f \,\mathrm{d}s \qquad \textit{and} \qquad t \mapsto \frac{1}{t}\int_0^t \mathrm{e}^{-s} M_s f \,\mathrm{d}s$$

converge locally uniformly to $\nu_0(f)\phi_0$ when t tends to infinity. Moreover if $w(x) = x^{r_1} + x^{r_2}$ with $r_1 < 1 < r_2$, then there exist constants $C \geqslant 1$ and $a > 0$ such that for all $f \in \mathcal{B}(w)$

$$\left\| \frac{1}{\log 2}\int_t^{t+\log 2} \mathrm{e}^{-s} M_s f \,\mathrm{d}s - \nu_0(f)\phi_0 \right\|_{\mathcal{B}(w)} \leqslant C\mathrm{e}^{-at} \left\| f - \nu_0(f)\phi_0 \right\|_{\mathcal{B}(w)}$$

and

$$\left\| \frac{1}{t}\int_0^t \mathrm{e}^{-s} M_s f \,\mathrm{d}s - \nu_0(f)\phi_0 \right\|_{\mathcal{B}(w)} \leqslant \frac{C}{t} \left\| f - \nu_0(f)\phi_0 \right\|_{\mathcal{B}(w)}.$$

Proof. Let $f \in \mathcal{C}_c^1(0,\infty)$. On the one hand, since $\mathrm{e}^{-t}M_t$ and R_t are contractions in $\dot{\mathcal{C}}$ and $\mathrm{e}^{-t}M_t f - R_t f$ tends to zero locally uniformly, we have by dominated convergence theorem the local uniform convergence

$$\frac{1}{\log 2}\int_t^{t+\log 2} \mathrm{e}^{-s} M_s f \,\mathrm{d}s - \frac{1}{\log 2}\int_t^{t+\log 2} R_s f \,\mathrm{d}s \xrightarrow[t\to\infty]{} 0.$$

On the other hand, due to the convergence

$$\left\| R_s f - \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N}\Big) \nu_k(f) \mathrm{e}^{\frac{2i\pi k}{\log 2}s} \phi_k \right\|_{\dot{\mathcal{B}}} \xrightarrow[N\to\infty]{} 0$$

we have that for all $t \geqslant 0$

$$\frac{1}{\log 2}\int_t^{t+\log 2} R_s f \mathrm{d}s = \nu_0(f)\phi_0.$$

This proves the local uniform convergence of the first integral of the lemma for $f \in \mathcal{C}_c^1(0,\infty)$, which remains valid for $f \in \dot{\mathcal{C}}_0$ by density. As a consequence the Cesàro means

$$\frac{1}{N}\sum_{n=0}^{N-1}\int_{n\log 2}^{(n+1)\log 2} \mathrm{e}^{-s} M_s f \,\mathrm{d}s = \frac{1}{N\log 2}\int_0^{N\log 2} \mathrm{e}^{-s} M_s f \,\mathrm{d}s$$

also converges to $\nu_0(f)\phi_0$ locally uniformly when $N \to \infty$, and it implies the convergence of the second mapping in the lemma. The uniform exponential convergence in weighted supremum norm follows from Theorem 11, integrating between t and $t + \log 2$, and the other one is obtained by integrating between 0 and t. □

The difference between the two speeds in the previous corollary can be interpreted as the difference in the amount of memory kept from the past.

5 Conclusion

In this work, we investigated how the cyclic asymptotic behavior of the rescaled solutions of Equation (1) exhibited in [8] is transposed in the measure setting. Despite the absence of Hilbert structure, we managed to build a suitable projection on the boundary spectral subspace by taking advantage of the general relative entropy of the dual equation. It allowed us to obtain the weak-* convergence of the rescaled measure solutions to a periodic behavior. Then, using Harris's ergodic theorem on time and space discrete sub-problems, we managed to get uniform exponential convergence in weighted total variation norm. To our knowledge no estimate on the speed of convergence was known before for such problems. Here we not only prove that the convergence takes place exponentially fast, but we also obtain explicit estimates on the spectral gap in terms of the division rate B.

In [38], more general growth rates than linear are considered, namely those satisfying $g(2x) = 2g(x)$. Our method would work in this case, replacing the weight x by the corresponding dual eigenfunction $\phi(x)$ and the space X by the functions such that $f(2x)/\phi(2x) = f(x)/\phi(x)$. However, considering such general coefficients, while interesting from mathematical point of view, is not motivated by modeling concerns, that is why we decided to focus on the linear case. In addition, it makes computations lighter, in particular those of the flow which is explicitly given by an exponential when $g(x) = x$.

Our method would also apply to more sophisticated models of mitosis. For instance the equation considered in [36] exhibits a similar countable family of boundary eigenelements for the singular mitosis kernel. To the prize of additional technicalities, our approach can be used to study its long time behavior.

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